THE CONTINUITY OF THE ROOTS OF AN ALGEBRAIC EQUATION

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The theorem that the roots of an algebraic equation whose term of highest degree has a non-vanishing coefficient are continuous functions of its coefficients is of fundamental importance, both in algebra and in geometry. The proofs usually given are rather long, and generally more closely allied in spirit to the theory of functions of a complex variable than to the elementary processes of algebra. For that reason a simple algebraic proof of this essential theorem seems to fill a real, if minute, gap in our algebraic theory. The object of this note is to give such a proof.

Theorem. *Given two algebraic equations of the same degree, so related that the coefficients of the first are constant and that of the highest power of the variable is not zero, while those of the second approach the corresponding coefficients of the first as limits; then the roots of the two equations, where each multiple root of order k is counted as k roots, may be put into such a one to one correspondence, that the absolute value of the difference of each two corresponding roots approaches zero.*

Let us write

$$f(x) \equiv a_0 x^n + \cdots + a_n \equiv a_0(x - \alpha_1) \cdots (x - \alpha_n),$$

$$\phi(x) \equiv (a_0 + \Delta a_0)\, x^n + \cdots a_n + \Delta a_n \equiv (a_0 + \Delta a_0)(x - \beta_1) \cdots (x - \beta_n).$$

We assume explicitly that $a_0 \neq 0$ and that all of the quantities Δa_i approach zero. In particular we assume that $|\Delta a_0|$ is so small that for that, and all smaller values $|a_0 + \Delta a_0| > 0$.

Let us now reduce the roots of the equation $\phi(x) = 0$ by α_1. We shall get a new equation $\psi(x) = 0$, the coefficient of x^n being $(a_0 + \Delta a_0)$ while the constant term is $\phi(\alpha_1)$. But since $f(\alpha_1) = 0$, we shall have

$$\phi(\alpha_1) = \Delta a_0\, \alpha_1^n + \cdots \Delta a_n.$$

If $|\alpha_1| = 1$, $|\phi(\alpha_1)| \leqq (n+1)|\Delta a_k|$, where Δa_k is the largest Δa in absolute value.

$$\text{If} \qquad |\alpha_1| \neq 1, \qquad |\phi(\alpha_1)| \leqq \left| \Delta a_k \frac{|\alpha_1^{n+1}| - 1}{|\alpha_1| - 1} \right|,$$

and this approaches zero as a limit. The limit of $a_0 + \Delta a_0$ is a_0, hence the limit of the product of the roots of ψ namely $\pm \phi(\alpha_1)/(a_0 + \Delta a_0)$, is zero; hence one root, at least of $\psi(x) = 0$, approaches zero as a limit.

Let us, therefore, write

$$\beta_1 = \alpha_1 + \Delta\alpha_1,$$

where $\Delta\alpha_1$ approaches zero.

$$a_0 x^n + \cdots a_n \equiv (x - \alpha_1)(a_0 x^{n-1} + b_1 x^{n-2} + \cdots b_{n-1})$$
$$\equiv (x - \alpha_1) f_1(x),$$

$$(a_0 + \Delta a_0)x_1^n + \cdots (a_n + \Delta a_n)$$
$$\equiv \Big(x - (\alpha_1 + \Delta\alpha_1)\Big)\Big((a_0 + \Delta a_0)x^{n-1} + c_1 x^{n-2} + \cdots c_{n-1}\Big)$$
$$\equiv \Big(x - (\alpha_1 + \Delta\alpha_1)\Big)\phi_1(x).$$

Multiplying out, and equating corresponding coefficients, we have

$$b_1 = a_0\alpha_1 + a_1, \qquad b_k = a_0\alpha_1^k + a_1\alpha_1^{k-1} + \cdots a_k,$$
$$c_1 = (a_0 + \Delta a_0)(\alpha_1 + \Delta\alpha_1) + (a_1 + \Delta a_1) = b_1 + \epsilon_1,$$
$$c_k = (a_0 + \Delta a_0)(\alpha_1 + \Delta\alpha_1)^k + \cdots (a_k + \Delta a_k) = b_k + \epsilon_k,$$

where ϵ_k approaches zero with the Δ's.

Hence one root of $\phi_1(x) = 0$ lies infinitesimally near one root of $f_1(x) = 0$, or a second root of $\phi(x) = 0$ approaches a second root of $f(x) = 0$ as a limit.

By a repetition of this process our theorem is proved.

The foregoing reasoning is, of course, invalid in the case where $a_0 = 0$. It is, however, very easy to treat this case in a similar manner.

Let us assume that $a_n \neq 0$, for this may always be effected by reducing all the roots by a constant quantity—a process that does not in the least alter the validity of the conclusion.

We then write two new equations:

$$F(x) \equiv a_n x^n + \cdots \qquad + a_0 = 0,$$

$$\Phi(x) \equiv (a_n + \Delta a_n)x^n + \cdots \qquad + (a_0 + \Delta a_0) = 0.$$

To be perfectly general we shall assume that the first p coefficients in $f(x) = 0$, or the last p in $F(x) = 0$ are zero. Then $F(x) = 0$ has p roots equal to zero, and the remaining roots are the reciprocals of the roots of $f(x) = 0$. The non-vanishing roots of $\Phi(x) = 0$ are the reciprocals of those of $\phi(x) = 0$. Applying our theorem to $F(x) = 0$ and $\Phi(x) = 0$ we have:

THEOREM. *If two equations be so related that the coefficients of the first are constants, while each coefficient of the second approaches the corresponding coefficient of the first as a limit, and if the coefficients of the p highest powers of the unknown in the first be zero, then if each root of multiplicity k be counted as k roots, the second equation will have p roots whose absolute value will increase beyond all limit, and the remaining roots of the second may be put into such a one to one correspondence with the roots of the first, that each root of the second will approach the corresponding root of the first as a limit.*

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